Interval Trees

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Outline

Interval Trees

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Interval Trees
A Problem
Solution
Implementation

Fenwick Trees
A Problem
Solution
Implementation

More Query/Update Problems
Using Transformations

Summary

1. Interval Trees
   - A Problem
   - Solution
   - Implementation
2. Fenwick Trees
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3. More Query/Update Problems
   - Using Transformations

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An Example Problem

A city has N buildings in a row, numbered from 1 to N. Initially, every building has height 0. Accept a sequence of queries and updates of the form

- Building i now has height h.
- What is the height of the tallest building in the range $[l, r]$?

Analysis: Naïve Solution

Simply store the height of each building:

- Each update requires $O(1)$ time
- Each query requires $O(N)$ time

Slightly Smarter Solution

- Divide city into “neighbourhoods” of $\sqrt{N}$ buildings
- Maintain the maximum height of each neighbourhood

Running time:

- Each update takes $O(\sqrt{N})$ time
- Each query takes $O(\sqrt{N})$ time (why?)

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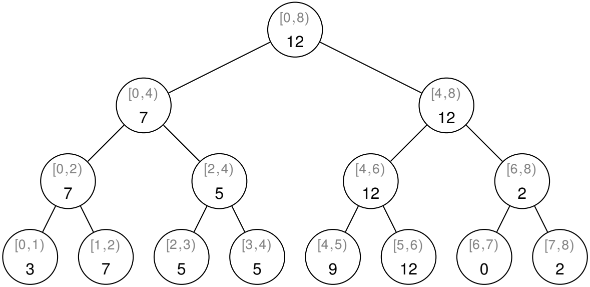
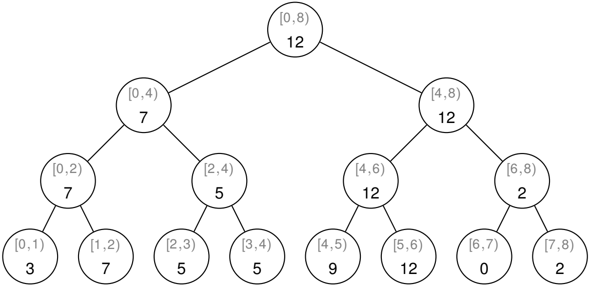
Interval Tree

Instead of just buildings and neighbourhoods, use a hierarchy:



Updates

Walk up the tree, updating ancestors



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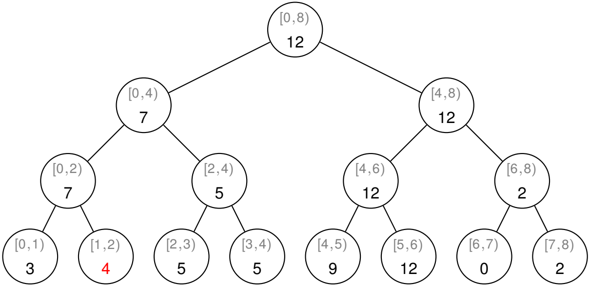

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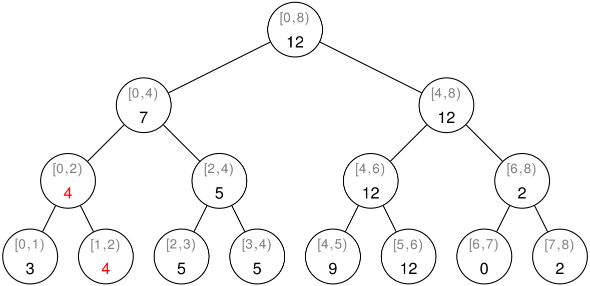

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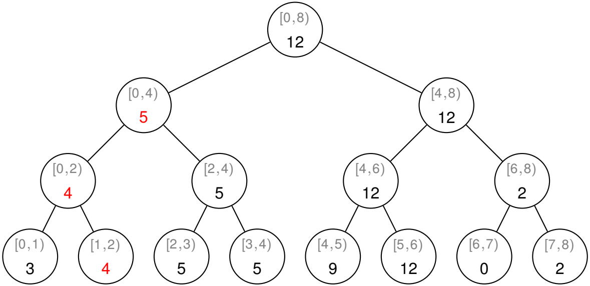
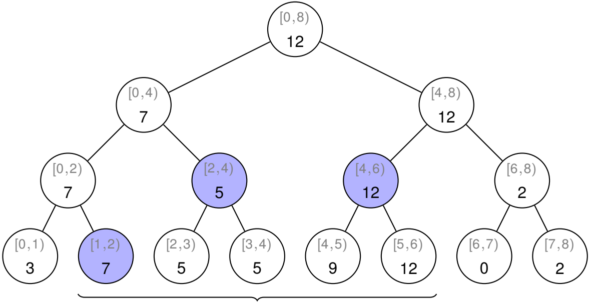


Queries

Pick a set of nodes to cover the range e.g. for $[1, 6)$:



Performance

- Each update touches $O(\log N)$ nodes
- Each query examines $O(\log N)$ nodes (why?)

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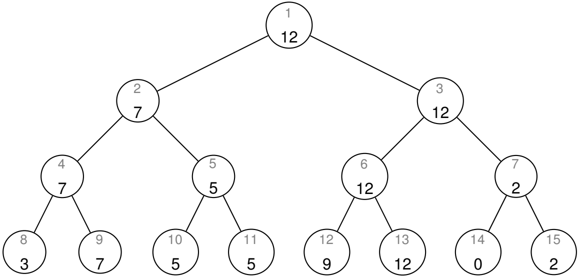
Representation

Number nodes in BFS order



- Parent of i is $\lfloor i/2 \rfloor$
- Children of i are $2i$, $2i + 1$
- Round up to a power of 2

Initialisation

```
void fix(int idx) {
  tree[idx] = max(tree[2 * idx],
                  tree[2 * idx + 1]);
}

vector<int> init(const vector<int> &values) {
  int bias = next_power2(values.size();
  vector<int> tree(2 * bias, 0);
  copy(values.begin(), values.end(),
      tree.begin() + bias);
  for (int i = bias; i > 0; i--)
    fix(tree, i);
  return tree;
}
```

Update

```
void update(int pos, int val) {
  pos += tree.size() / 2;
  tree[pos] = val;
  for (pos = pos / 2; pos > 0; pos = pos / 2)
    fix(tree, pos);
```

Query

```
int query(int L, int R) {
  int ans = 0;
  L += bias; R += bias;
  while (L < R) {
    if (L & 1) {
      ans = max(ans, tree[L]);
      L++;
    }
    if (R & 1) {
      R--;
      ans = max(ans, tree[R]);
    }
    L /= 2; R /= 2;
  }
  return ans;
}
```

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A city has N buildings in a row, numbered from 1 to N. Initially, every building has height 0. Accept a sequence of queries and updates of the form

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- What is the sum of the building heights in the range $[l, r]$?

An Example Problem

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- Building i now has height h.
- What is the sum of the building heights in the range $[l, r]$?

You only have enough memory for $N + \epsilon$ integers.

A Non-Obvious Solution

Store a prefix sum of the heights: sum of the first i heights for every i.

Query Take the difference between two prefix sums: $O(1)$

Update Modify all prefix sums that include this element: $O(N)$

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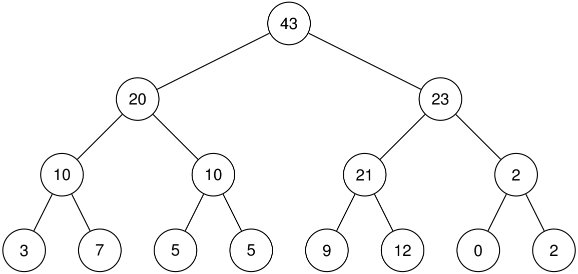
Interval Tree is Redundant



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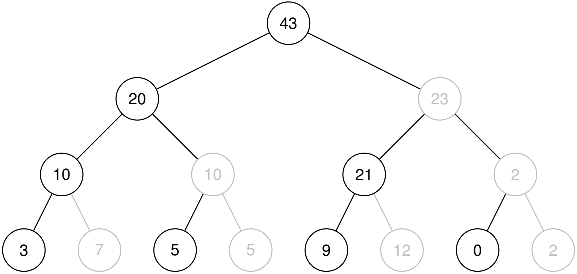

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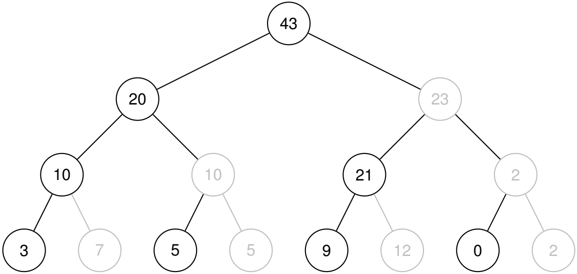
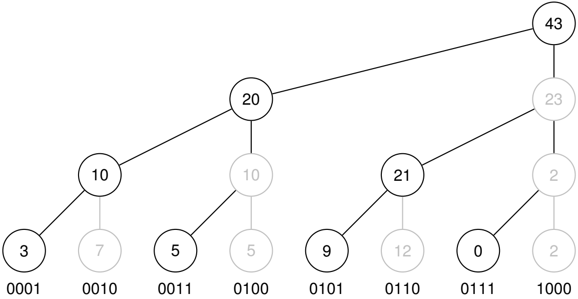


These nodes are not involved in prefix sum queries.

Representation



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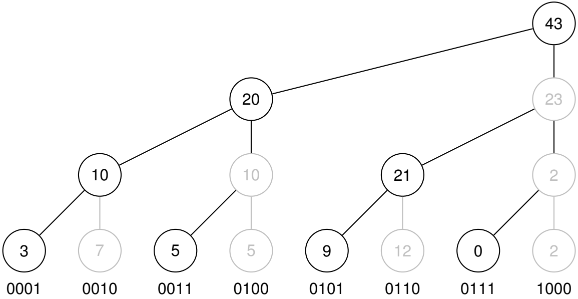

Element i is sum of 2^k elements, $2^k \mid i$, k is maximum

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Finding The Parent

The parent of i is $i + 2^k$ where $2^k \mid i$, k is maximal.

Example:

$$\begin{array}{r} 11001000 \\ +\ 00001000 \\ =\ 11010000 \end{array}$$

Finding The Parent

The parent of i is $i + 2^k$ where $2^k \mid i$, k is maximal.
Example:

$$\begin{array}{r} \texttt{11001000} \\ + \texttt{00001000} \\ = \texttt{11010000} \end{array}$$

To find 2^k, we take i and mask off $i - 1$:

$$\begin{array}{r} \texttt{11001000} \\ \&\ \texttt{\~{}11000111} \\ = \texttt{00001000} \end{array}$$

Update

```
void bit_add(int *bit, int p, int v) {
  while (p < size) {
    bit[p] += v;
    p += p & ~(p - 1);
  }
}
```

Query

To query a prefix sum, we add the current node, then see what is left.

```
int bit_query(const int *bit, int p) {
  int ans = 0;
  while (p > 0) {
    ans += bit[p];
    p &= p - 1; // same as p -= p & ~(p - 1);
  }
  return ans;
}
```

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Range Update, Point Query

Starting with an array a, handle the following queries

- Update: increment by h across a range $[l, r]$
- Query: return a_i

Range Update, Point Query

Solution

Operate on array of adjacent differences instead:

$$b_1 = a_1, b_i = a_i - a_{i-1}$$

Range Update, Point Query

Solution

Operate on array of adjacent differences instead:

$$b_1 = a_1 \, , b_i = a_i - a_{i-1}$$

Operations become:

Update $b_l \leftarrow b_l + h$, $b_{r+1} \leftarrow b_{r+1} - h$

Range Update, Point Query

Solution

Operate on array of adjacent differences instead:

$$b_1 = a_1, b_i = a_i - a_{i-1}$$

Operations become:

Update $b_l \leftarrow b_l + h$, $b_{r+1} \leftarrow b_{r+1} - h$

Query Return $a_i = \sum_1^i b_j$ using Fenwick tree.

Range Update, Range Query

Starting with an array a, handle the following queries

- Update: increment by h across a range $[l, r]$
- Query: return the sum $\sum_{i=l}^{r} a_i$

Note: sufficient to be able to answer $\sum_{i=1}^{r} a_i$.

Range Update, Range Query
Solution

Start with the same transformation as before:

$$b_1 = a_1, b_i = a_i - a_{i-1}$$

Query is

$$\begin{aligned}\sum_{i=1}^{r} a_i &= \sum_{i=1}^{r}\sum_{j=1}^{i} b_j \\ &= \sum_{i=1}^{r}(r - 1 - i)b_i \\ &= (r-1)\left(\sum_{i=1}^{r} b_i\right) - \left(\sum_{i=1}^{r} ib_i\right)\end{aligned}$$

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Let $c_i = ib_i$. Then we need Fenwick trees for b and c.

Summary

- Interval trees are a general-purpose tool for accelerating operations on ranges.
- Fenwick trees are less general, but more compact and easier to implement.
- Both have relatively low overhead and simple implementation due to the implicit tree structure.