Range queries and Fenwick Trees

Version 1.1

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Preliminaries

- All ranges will be half open ranges $e \in [a, b) \iff a \leq e < b$
- Occasionally 1 is a more convenient starting index than 0

Susie has questions

Problem

Susie has $1 < N < 10^6$ model ships arranged in a sequence numbered $0, \ldots, N-1$. The ith boat has a size of s_i ($1 < s_i < 10^9$). At any given time Susie may replace a boat with another boat of a different size. Given two integers a and b, report the sizes of all the ships between a and b.

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In summary

- $\sim 10^6$ model ships of different sizes $\sim 10^9$.
- Susie can change the size of a ship.
- Report all sizes of ships between a and b.

Susie's questions are easy to answer

Solution

Store an an array of all the ships.

Time Complexity

- Let $m = b - a$. m is the width of the query.
- O(N) construction
- O(m) query
- $O(1)$ update

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Observations

- The min function is associative i.e. $\min(a, \min(b, c)) = \min(\min(a, b), c)$
- In other words, min function forms a semigroup with the integers

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- In other words, min function forms a semigroup with the integers
- It is unnecessary to iterate over m since

  $$\min(x_1, x_2, \ldots, x_{2n}) = \min(\min(x_1, \ldots, x_n), \min(x_{n+1}, \ldots, x_{2n}))$$

  allows us to "cache" queries.
- We can query in better than $O(m)$ time.

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- Perfectly balanced binary tree.
- The leaf nodes correspond with s_i.
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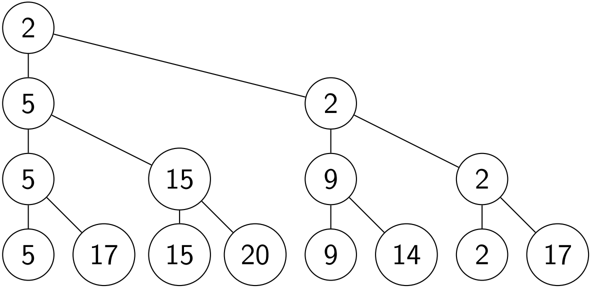
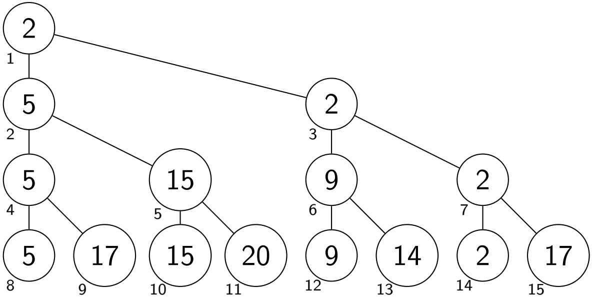


Representing a Perfectly Balanced Binary Tree

- Represent the tree as an array indexed from 1
- For every index i the
  - left child is $2i$
  - right child is $2i + 1$



Update by walking up the tree

```
def update(index, value):
    index += N
    seg_tree[index] = value
    index /= 2
    while index > 0:
    seg_tree[index] = min(
            seg_tree[2 * index],
            seg_tree[2 * index + 1]
        )
        index /= 2
```

Query by walking up the tree

```
def query(a, b):
    a += N
    b += N
    ans = ∞
    while a < b:
        if a % 2 == 1:
            ans = min(seg_tree[a], ans)
            a += 1
        if (b - 1) % 2 == 0:
            ans = min(seg_tree[b - 1], ans)
        a /= 2
        b /= 2
```

Time complexity

- $O(N)$ construction
- $O(\log N)$ updates
- $O(\log N)$ query

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Solution

When updating a range, if a node is completely within the range, mark it as overridden and don't update the children.

Update code

```
def rec_update(i, l, r, v):
    a = segment_start(i)
    b = segment_end(i)
    if l <= a and b <= r:
        # Completely contained in the interval
        overide[i] = True
        seg_tree[i] = v
    elif l < b and a < r:
        # Intersects, thus update children
        push_down_overide(i)
        rec_update(2 * i, l, r, v)
        rec_update(2 * i + 1, l, r, v)
        seg_tree[i] = min(seg_tree[2 * i], seg_tree[2 * i + 1])

def push_down_overide(i):
    l = 2 * i
    r = l + 1
    if overide[i]:
        overide[i] = False
        overide[l] = overide[r] = True
        seg_tree[l] = seg_tree[r] = seg_tree[i]
```

Query

```
def query(i, l, r):
  a = segment_start(i)
  b = segment_end(i)
  if l <= a and b <= r:
      # Completely contained in the interval
      return seg_tree[i]
  elif b <= l or r <= a:
      # Don't intersect do nothing
      return ∞ # Return identity
  else:
      push_down_overide(i)
      return min(query(2 * i, l, r), query(2 * i + 1, l, r))
```

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Find the sum of the sizes of the boats between a and b. (Only updating single points at a time).

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Observation

- Addition has an identity (0)
- and an inverse operation ($-$)
- Addition forms a group with the integers

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- and an inverse operation ($-$)
- Addition forms a group with the integers

We can subtract!

Prefix sums

```
prefix_sum = [0]
for i in range(N):
    prefix_sum.append(ships[i] + prefix_sum[-1])

def query(l, r):
    return prefix_sum[r] - prefix_sum[l]
```

"Subtraction" is required

Prefix sums

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Time Complexity

- O(N) construction
- O(1) query
- O(N) update

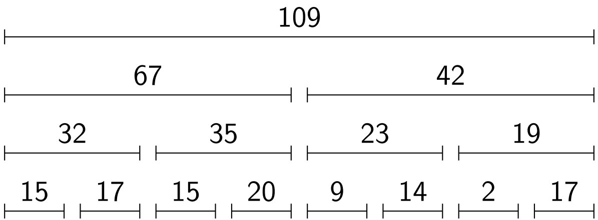
Update is too slow!

Fenwick trees

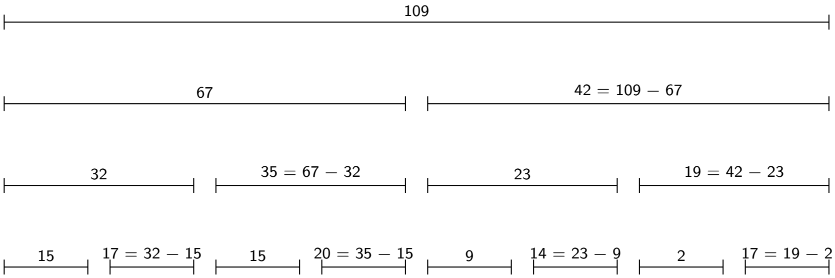
Ideas

- We can use a segment tree, but we can do better

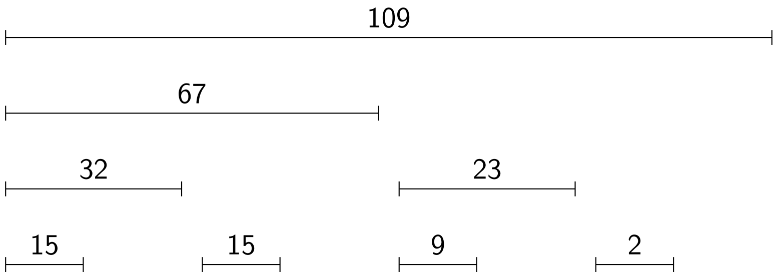
Combine the prefix sum with the segment tree



Right nodes are redundant



Chop off the right nodes



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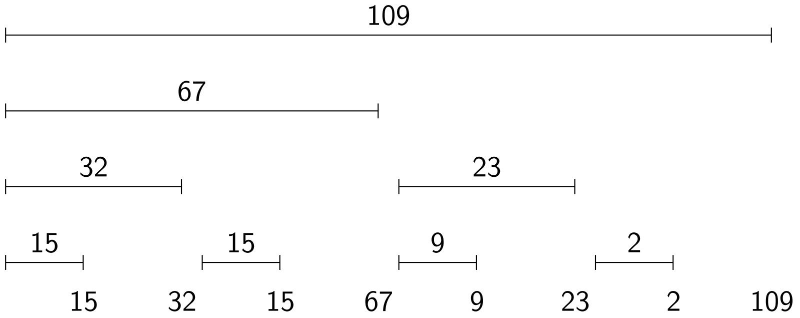
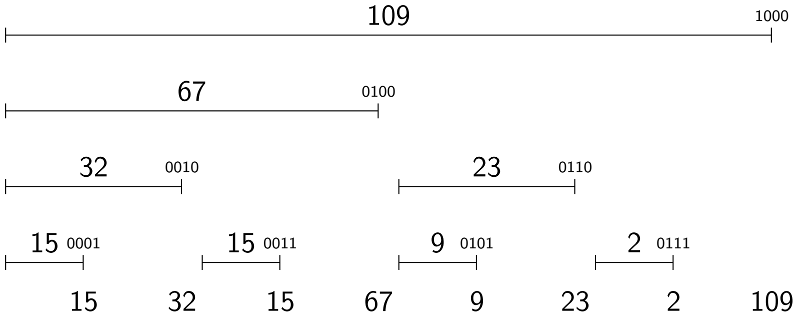


We are left with N numbers



Storage

- We only have N nodes (not $2N$)
- We use an array indexed from 1.
- Let s be the greatest power of 2 that divides i
- Index i contains the sum of $[i - r + 1, i + 1)$

Updating

- We update by increasing rather than setting.
- It is easy to compute what to increase
- i is the smallest index that contains s_i
- $i + r$ is the next element that contains i

Computing r

We can compute the largest power of two by using `i & ~(i - 1)`

```
   10101000
-         1
-----------
~  10100111
-----------
   01011000
&  10101000
-----------
   00001000
```

Code for fenwick tree

```python
def update(i, v):
    while i < N:
        fenwick_tree[i] += v
        # Go to parent
        i += (i & ~(i - 1))

def query(i):
    acc = 0 # Identity
    while i > 0:
        acc += fenwick_tree[i]
        # Go to previous
        i -= (i & ~(i - 1))
```

query(a, b) $=$ query(b) - query(a)

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We can apply a tranformation.

- $d_i = s_i - s_{i-1}$
- $d_0 = s_0$

Construct a fenwick tree over d

- We can query a point just by querying `query(point)`
- Update a range by update(a, v) and update(b, -v)

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Construct a fenwick tree over d

- We can query a point just by querying `query(point)`
- Update a range by update(a, v) and update(b, -v)
- Beware of off-by-one errors

Susie wants to query a range

$$
\begin{aligned}
\sum_{i=0}^{a-1} s_i &= \sum_{i=0}^{a-1} \sum_{j}^{i} d_j \\
&= \sum_{i=0}^{a-1} (a-i) d_j \\
&= a \left(\sum_{i=0}^{a-1} d_i \right) - \left(\sum_{i=0}^{a-1} i d_i \right)
\end{aligned}
$$

Make a Fenwick tree with id_i as well.

Better solution

Use a segment tree instead